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3D INVERSE SCATTERING

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Abstract—We give a self-contained presentation of the method developed by R. Newton for solving the 3D inverse scattering problem and give an analysis of what is proved and what is not proved in this theory.

1. INTRODUCTION

There is wide interest in the inverse problem for the Schödinger equation in three dimensions: given $A(\theta', \theta, k)$, the amplitude for a plane wave with wavenumber k incident on a potential from the direction θ to be elastically scattered into direction θ', find the potential $q(x)$ that caused the scattering. More generally, given a function $A(\theta', \theta, k)$, determine whether or not it *is* a scattering amplitude for a potential $q(x)$, construct $q(x)$ and its scattering amplitude A_q, and finally, prove that $A_q = A$, the original function. Much work has been done in this area by many authors (L. Faddeev, R. Newton, M. Ablowitz, R. Beals, R. Coifman, A.G. Ramm, G. Henkin, R. Novikov, and others). The problem, however, is far from being well understood. There are, roughly speaking, three approaches to the problem. The first, based on earlier work by Faddeev, Beals and Coifman, and others is reviewed in [1]. The second, developed by R. Newton, is based on a Marchenko–type equation and is discussed in [2]. The third, developed by the first author [3], is based on the fundamental S-matrix relation discussed below, and a variant is used in [4]. We discuss the second approach in this paper, providing a commentary on the approach developed by Newton.

Newton's solution [2], apart from certain technical assumptions, is this: given $A(\theta', \theta, k)$ construct an integral equation (M). If (M) has a unique solution, and if the solution satisfies a compatibility condition (the "miracle"), then a potential q for a Schrödinger equation can be constructed; if, in addition, another integral equation $(M\#)$ (which is Equation (2.31) in [2] with the right-hand side equal to zero) has only the trivial solution, then $A = A_q$. Conversely, if we know that $A = A_q$, then Newton claims [5] that Equation (M) does have a unique solution, this solution does satisfy the miracle condition, and the Equation $(M\#)$ also has only the trivial solution. Unfortunately some of the proofs (in particular, Lemmas 2.4.1 and 2.4.6) of [2] are incorrect as originally published (though 2.4.1. is corrected in an errata sheet [6]), and we have found a counter example (Section 4.2. below) to a crucial uniqueness claim used in [5, p. 2423]. (This counter example was communicated to Professor Newton and he withdrew [7] where Lemma 2 is a variant of the above uniqueness result.) We are unable to prove the uniqueness of the solution to (M) and $(M\#)$ if $A = A_q$, so the necessity of these uniqueness conditions is an important open problem, but on the assumption that these conditions hold, we give a self-contained account of Newton's approach. However, in place of $(M\#)$, we use (M'), Equation (2.51) below. We

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX often include more details of the calculations than are found in existing publications so that the methods will be clear to outsiders in this field and we provide motivation for our assumptions before we make them explicitly. Our derivations are sometimes new (these are pointed out in the text).

There is a technical assumption we make at the outset that is stronger than Newton usually uses: the class Q of functions which contains the potential $q(x)$ which we either start from (Section 2) or reconstruct (Section 3) is the class of potentials which has no bound states, are smooth (at least $\mathbb{C}^1(\mathbf{R}^3)$), have compact support and for which $k = 0$ is neither a bound state nor a resonance. The condition that q has compact support is to some extent necessary if one wishes to treat the inverse problem with noisy data: it is not possible to recover the tail of a potential which is smaller in absolute value than the noise.

Very little is known about conditions on $A_q(\theta', \theta, k)$, which would ensure that $q(x)$ have *any* special property, much less that it belong to the above class. Finding such conditions is an important problem, and we summarize some of what is known about it in Section 4.3.

Here is an outline of the contents of this paper: In Section 2, we discuss the scattering solution to the Schrödinger equation and the scattering amplitude, and obtain the basic integral Equations (M), (M') and (F). In Section 3, we explain how to construct the potential and the Schrödinger equation directly from the solution to (M), and under what assumptions this can be done. Then we explain the extra assumptions needed to prove that the function A used in equation (M) is equal to A_q, the scattering amplitude for the constructed potential. This is a crucial point which is necessary to discuss in studying this inverse problem: it allows one to close the loop which starts from the given data A, produces a potential q which in turn produces new data A_q, which (we hope) equals the input data: $A \to q \to A_q = A$. In Section 4.1 we compare the approach of Newton with that of Ramm [4,8]; in 4.2 we give the counter example mentioned above; in 4.3 we discuss some of what is known about the relations between A_q and q. In Section 5 we state two theorems that summarize the contents of this paper.

2. THE S-MATRIX RELATION AND ITS CONSEQUENCES

This section gives the physical motivation for the definitions and functions we use, and for the equations we study rigorously in Section 3.

2.1. The Scattering Solution

The scattering solutions to the Schrödinger equation with potential q

$$\left(-\nabla^2 + q(x)\right) u = k^2 u, \tag{2.1}$$

obey the outgoing boundary condition

$$u^+(x, \theta, k) = e^{ik\theta\cdot x} + v(x, \theta, k), \tag{2.2}$$

$\theta \in S^2$ is a unit vector that specifies the direction of an incident plane wave with wavenumber $k > 0$, and v satisfies the outgoing radiation condition

$$\lim_{r\to\infty} r\left(ikv - \frac{\partial v}{\partial r}\right) = 0, \quad r = |x|, \tag{2.3}$$

uniformly in $\theta' = x/r \in S^2$. The system (2.1)–(2.3) has a unique solution for q in some specified classes. Important classes are potentials with support in a ball: $q \in Q_a$,

$$Q_a = \{q : q = \bar{q},\, q \in L^2(B_a),\, q = 0 \quad \text{outside}\, B_a\},\ B_a = \{x : |x| \le a\}; \tag{2.4}$$

and potentials that decrease at infinity: $q \in Q(\beta)$,

$$Q(\beta) = \{q : q = \bar{q},\, q \in L^2_{\text{loc}},\, |q(x)|(1 + |x|)^\beta \le c \quad \text{for}\, |x| > R,\, \beta > 2\}, \tag{2.5}$$

where $R > 0$ is an arbitrary large number.

For potentials in such classes, in particular, for those in class Q of the Introduction,

$$v(x,\theta,k) = A(\theta',\theta,k)\,\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \theta' = x/r, \quad r \to \infty. \tag{2.6}$$

The function $A(\theta',\theta,k)$ is the scattering amplitude corresponding to the potential q. It is determined uniquely by q and, to emphasize this dependence, we sometimes write $A_q(\theta',\theta,k)$.

Equations (2.1)–(2.3) can be rewritten as the integral equation

$$u^+(x,\theta,k) = e^{ik\theta\cdot x} - \int\limits_{R^3} \frac{e^{ik|x-y|}}{4\pi|x-y|}\, q(y)u^+(y,\theta,k)\,dy, \tag{2.7}$$

and inspecting the behavior for large $|x|$ gives a formula for A_q:

$$A_q(\theta',\theta,k) = -\frac{1}{4\pi}\int\limits_{R^3} e^{-ik\theta'\cdot y}\, q(y)u^+(y,\theta,k)\,dy. \tag{2.8}$$

For potentials in Q (and wider classes, in particular for $q \in Q(\beta), \beta > 2$) Equation (2.7) implies that $e^{-ik\theta\cdot x}u^+(x,\theta,k)$ is meromorphic for $k \in \mathbf{C}^+$, and is uniformly bounded in $x \in \mathbf{R}^3$, $\theta \in S^2$ for $\operatorname{Im} k >> 1$. Thus, in particular, for large real k we have

$$u^+(x,\theta,k) = e^{ik\theta\cdot x} + o(1), \quad k \to \infty, \tag{2.9}$$

and using this in (2.8) gives the asymptotic behavior:

$$A_q(\theta',\theta,k) = -\frac{1}{4\pi}\int\limits_{R^3} e^{-ik(\theta'-\theta)\cdot x} q(x)\,dx + o(1), \quad k \to \infty. \tag{2.10}$$

The integral on the right of (2.10) is called the Born approximation to $A_q(\theta',\theta,k)$. If we fix an arbitrary $p \in \mathbf{R}^3$, choose k, θ', and θ, such that $k(\theta'-\theta) = p$ and let $k \to \infty$, we find the exact result

$$-4\pi \lim_{\substack{k\to\infty \\ k(\theta'-\theta)=p}} A_q(\theta',\theta,k) = \tilde{q}(p) := \int\limits_{\mathbf{R}^3} e^{-ip\cdot x} q(x)dx, \tag{2.11}$$

where $\tilde{q}(p)$ is the Fourier transform of the potential.

It is important to notice what (2.11) does and does not show:

(a) The scattering amplitude $A_q(\theta',\theta,k)$ does uniquely determine the potential $q(x)$. (This is true even if we only know $A(\theta',\theta,k)$ on a sequence $\{\theta'_n, \theta_n, k_n\}$ (which depends on p) that permits evaluation of the limit in (2.11)). Thus, if we know in advance that a function $A(\theta',\theta,k)$ is in fact the scattering amplitude corresponding to some unknown potential q in Q, then we may correctly use (2.11) to recover the potential.

(b) If we do *not* know in advance that the function $A(\theta',\theta,k)$ comes from a potential (for example, $A(\theta',\theta,k)$ might be $A_q(\theta',\theta,k) + \epsilon(\theta',\theta,k)$ with ϵ a noise term; or $A(\theta',\theta,k)$ might be an analytic approximation to a scattering amplitude given numerically at a limited set of $\theta',\theta,k)$ then, although the limit in (2.11) may exist and define a potential $q(x)$, it is wrong to infer that $A = A_q$. For example, we may change the given function A in an arbitrary way for $k < k_0$, where $k_0 > 0$ is any fixed number, and still discover the same $q(x)$ because (2.11) is completely insensitive to the value of A at any finite k. Thus, in a practical problem, even if we know $A_q(\theta',\theta,k)$ exactly for a finite range of k, Equation (2.11) is not a directly useful inversion procedure. It is usually also pointed out that measuring the scattering amplitude in a neighborhood of $\theta = \theta'$ at large k is a very difficult experimental problem.

These observations suggest two important problems.

THE CHARACTERIZATION PROBLEM. Give necessary and sufficient conditions that a function $A(\theta',\theta,k)$ must satisfy in order that it be the scattering amplitude corresponding to a local potential $q(x)$ belonging to a certain class, e.g., Q, Q_a or $Q(\beta)$.

THE INVERSION PROBLEM. Give procedures for finding $q(x)$ that use $A(\theta',\theta,k)$ for all values of k, not only for $k \to \infty$.

2.2. The S-Matrix

The outgoing wave solutions u^+, discussed in Section 2.1, are physically important for the following reason. Far from the scattering center, a localized solution of the time-dependent Schrödinger equation is a superposition of $e^{-ik^2t}u^+(x,\theta,k)$ for θ, k near some fixed direction θ_0 and wavenumber $k_0 > 0$. That is

$$\psi(x,t) \sim \int d^3K\chi(K)\left\{e^{-ik^2t+ik\theta\cdot x} + A(\theta',\theta,k)\frac{e^{ikr-ik^2t}}{r}\right\}, \tag{2.12}$$

with $\chi(K)$ a smooth function with support in a neighborhood of $K = K_0$, $K = (k,\theta)$, $K_0 = (k_0,\theta_0)$. When $t \to -\infty$ only the plane wave has a point of stationary phase to contribute to the integral; for $t \to +\infty$ both the plane wave and the spherical wave terms can contribute. Thus, $u^+(x,\theta,k)$ represents an initial plane wave moving toward the potential; the wave evolves in time into the same plane wave plus a spherical wave moving away from the scatterer. Thus, it is plausible that the set $\{u^+(x,\theta,k) : \theta \in S^2, k > 0\}$ can be used to describe a physically reasonable scattering experiment.

There is another set of solutions to the Schrödinger Equation (2.1). These obey incoming wave boundary conditions

$$u^-(x,\theta,k) = e^{ik\theta\cdot x} + w(x,\theta,k),\ \theta \in S^2, \quad k > 0, \tag{2.13}$$

with

$$\lim_{r\to\infty} r\left(ikw + \frac{\partial w}{\partial r}\right) = 0. \tag{2.14}$$

The same sort of argument as above shows that these represent, for negative times, a plane wave moving toward the scattering center superimposed with a spherical wave collapsing in on the center. This wave evolves into a plane wave moving away from the scatterer. Thus, the u^- are time-reversed forms of u^+, and any function that can be formed from the u^+ can be formed from the u^- as well. It is therefore physically plausible, and can be proved for $q \in Q$, that there is a unitary (in $L^2(S^2)$) operator S connecting the two sets:

$$u^+(x,\theta,k) = \int_{S^2} d\theta' u^-(x,\theta',k)\, S(\theta',\theta,k). \tag{2.15}$$

Notice that the physical interpretation requires $k > 0$ in both u^+ and u^-. Now we extend u^+ to $k < 0$. To do so, consider the function

$$\phi(x,\theta,k) = u^-(x,-\theta,-k), \quad \text{for} \quad k < 0, \tag{2.16}$$

so the argument $-k$ of u^- is positive. This function obeys the Schödinger equation and the boundary condition

$$\phi(x,\theta,k) = e^{ik\theta\cdot x} + z(x,\theta,k), \tag{2.17}$$

with

$$\lim_{r\to\infty} r\left(-ikz + \frac{\partial z}{\partial r}\right) = 0, \quad k < 0. \tag{2.18}$$

This is the *outgoing* radiation condition (2.3) and so $\phi(x,\theta,k)$ and $u^+(x,\theta,k)$, $k < 0$, obey the same equation and boundary conditions (2.1)–(2.3) and are thus the same function. This extends the definition of u^+ to negative k:

$$u^+(x,\theta,k) = u^-(x,-\theta,-k), \quad k < 0. \tag{2.19}$$

With this extended definition of $u^+(x,\theta,k)$, we can rewrite the basic S-matrix relation in terms of u^+, which we call simply u from now on:

$$u(x,\theta,k) := u^+(x,\theta,k), \quad \forall\, k \in \mathbf{R}; \tag{2.20}$$

$$u(x,\theta,k) = \int_{S^2} d\theta' S(\theta',\theta,k)\, u(x,-\theta',-k), \quad \forall\, k. \tag{2.21}$$

2.3 Properties of the Scattering Amplitude

The behavior of the scattering solution $u(x,\theta,k)$ for large $r=|x|$,

$$u(x,\theta,k)=e^{ik\theta\cdot x}+A(\theta',\theta,k)\frac{e^{ikr}}{r}+o\left(\frac{1}{r}\right),\ r\to\infty,\theta'=x/r, \tag{2.22}$$

determines the scattering amplitude $A(\theta',\theta,k)$ and we establish some of its properties.

First, if $q(x)=\bar{q}(x)$ then $u(x,\theta,k)$ and $\overline{u(x,\theta,-k)}$ obey the same equation and boundary condition at infinity and so are identical. Thus,

$$A(\theta',\theta,k)=\overline{A(\theta',\theta,-k)}. \tag{2.23}$$

Notice that the values of $A(\theta',\theta,k)$ for all k are determined from its values in the physically accessible region $k>0$. We call (2.23) "the reality property."

Second, a formula closely related to time-reversal is that the amplitudes for the inverse processes $k,\theta\to k,\theta'$ and $k,-\theta'\to k,-\theta$ are equal:

$$A(\theta',\theta,k)=A(-\theta,-\theta',k). \tag{2.24}$$

We call (2.24) "reciprocity."

Third, the S-matrix relation (2.21) evaluted for large r yields (we integrate over θ'' and use $\theta'=x/r$ as before)

$$e^{ik\theta\cdot x}+A(\theta',\theta,k)\frac{e^{ikr}}{r}=\int_{S^2}d\theta''S(\theta'',\theta,k),\left[e^{ik\theta''\cdot x}+A(\theta',-\theta'',-k)\frac{e^{-ikr}}{r}\right]. \tag{2.25}$$

It is convenient to write

$$S(\theta'',\theta,k)=\delta(\theta''-\theta)+T(\theta'',\theta,k), \tag{2.26}$$

for then the plane wave terms on both sides of (2.25) cancel. Then, for once differentiable T, one has:

$$\int_{S^2}d\theta''e^{ik\theta''\cdot x}T(\theta'',\theta,k)=\frac{2\pi}{ikr}\left[T(\theta',\theta,k)\,e^{ikr}-T(-\theta',\theta,k)\,e^{-ikr}\right]+o\left(\frac{1}{r}\right),\theta'=x/r, \tag{2.27}$$

and equating coefficients of $\frac{e^{ikr}}{r}$ gives

$$\frac{ik}{2\pi}A(\theta',\theta,k)=T(\theta',\theta,k). \tag{2.28}$$

Equating coefficients of $\frac{e^{-ikr}}{r}$ gives (using (2.23), the reality property)

$$A(-\theta',\theta,k)-\overline{A(\theta',-\theta,k)}=\frac{ik}{2\pi}\int_{S^2}d\theta''\,\overline{A(\theta',-\theta'',k)}\,A(\theta'',\theta,k), \tag{2.29}$$

and using reciprocity and replacing $\theta'\to-\theta'$ gives

$$A(\theta',\theta,k)-\overline{A(\theta,\theta',k)}=\frac{ik}{2\pi}\int_{S^2}d\theta''\,\overline{A(\theta'',\theta',k)}\,A(\theta'',\theta,k). \tag{2.30}$$

This is called "unitarity" or the "generalized optical theorem."

Summarizing: the classical properties of $A(\theta', \theta, k)$ are

$$A(\theta', \theta, k) = \overline{A(\theta', \theta, -k)} \quad \text{Reality,} \tag{2.31}$$
$$A(\theta', \theta, k) = A(-\theta, -\theta', k) \quad \text{Reciprocity,} \tag{2.32}$$
$$A(\theta', \theta, k) - \overline{A(\theta, \theta', k)} = \frac{ik}{2\pi} \int_{S^2} d\theta'' \, \overline{A(\theta'', \theta', k)} \, A(\theta'', \theta, k) \quad \text{Unitarity,} \tag{2.33}$$
$$S(\theta', \theta, k) = \delta(\theta' - \theta) + \frac{ik}{2\pi} A(\theta', \theta, k). \tag{2.34}$$

These conditions of reality, reciprocity and unitarity are the classical conditions on A (or S, via Equation (2.34)). They are necessary conditions that A must satisfy in order to be a scattering amplitude for a real $q \in Q$, or, for brevity, they are necessary conditions on A for A to be ***admissible***. These conditions are not sufficient. For example,

$$A(\theta', \theta, k) = \frac{1}{c - ik}, \tag{2.35}$$

with c a real constant, satisfies the classical conditions but does not correspond to any potential in the class Q (or $Q(\beta)$). This is easily seen from the following argument. If (2.35) were a scattering amplitude for a q for which the Born inversion formula (2.11) holds, then formula (2.11) would imply that $\tilde{q}(p) = 0$, so that $q(x) = 0$. This implies that $A_q(\theta', \theta, k) = 0$, a contradiction. Thus, the function (2.35) cannot be the scattering amplitude for any $q(x)$ for which formula (2.11) holds. (It *is* the scattering amplitude for a "zero range" potential [8].)

2.4 The Basic Integral Equations

There are a variety of necessary conditions and a variety of sufficient conditions for $A(\theta', \theta, k)$ to be the scattering amplitude for a local potential, that is, for A to be admissible. These have been given by Faddeev, Newton, and Ablowitz and Nachman, among others.

Conditions on A that are necessary and sufficient for A to be a scattering amplitude for a potential in some class (that is, characterizations of the class of scattering amplitudes) have been found by Ramm [3], Ramm and Weaver [4], Henkin and Novikov [1], and Newton [5]. We will discuss the relative merits of some of these conditions in Section 4. The characterization which we describe in this paper is based on equations which are required to have unique solutions satisfying certain conditions. We now derive these equations which were originally derived by Newton [2].

The starting point is the S-matrix relation (2.21),

$$\begin{aligned} u(x, \theta, k) &= \int_{S^2} d\theta' S(\theta', \theta, k) \, u(x, -\theta', -k) \\ &= u(x, -\theta, -k) + \frac{ik}{2\pi} \int_{S^2} d\theta' A(\theta', \theta, k) \, u(x, -\theta', -k), \end{aligned} \tag{2.36}$$

where u is the scattering solution to the equation

$$(\nabla^2 + k^2) \, u(x, \theta, k) = q(x) \, u(x, \theta, k), \tag{2.37}$$

with the asymptotic behavior

$$u(x, \theta, k) = e^{ik\theta \cdot x} + A(\theta', \theta, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \; r \to \infty, \; \theta' = x/r. \tag{2.38}$$

Introduce the function

$$\nu(x, \theta, k) = e^{-ik\theta \cdot x} u(x, \theta, k) - 1 \tag{2.39}$$

into (2.36). Notice that $\nu(x,\theta,k)$ vanishes as $|x| \to \infty$ and that, for any fixed x, it is $o(1/k)$ if q is sufficiently well behaved (say $C^1(B_a)\bigcap Q_a$). The $o(1/k)$ decay guarantees that we can take the Fourier transform of $\nu(x,\theta,k)$ in the k variable in $L^2(\mathbf{R})$, but we will not try to get the least restrictive assumptions concerning $q(x)$ in this paper. Our objective is to present the basic ideas of inverse scattering theory in a simple way. When (2.39) is used in (2.36), we find

$$\begin{aligned}\nu(x,\theta,k) = \nu(x,-\theta,-k) + \frac{ik}{2\pi}\int_{S^2} d\theta' A(\theta',\theta,k)\, e^{ik(\theta'-\theta)\cdot x}\, \nu(x,-\theta',-k)\\ + \int_{S^2} d\theta' \frac{ik}{2\pi} A(\theta',\theta,k)\, e^{ik(\theta'-\theta)\cdot x}.\end{aligned} \tag{2.40}$$

Take the Fourier transform in k of (2.40), define

$$\eta(x,\theta,\alpha) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik\alpha}\, \nu(x,\theta,k), \tag{2.41}$$

$$M(x,\theta',\theta,\alpha) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik\alpha}\, e^{ik(\theta'-\theta)\cdot x} \frac{ik}{2\pi} A(\theta',\theta,k), \tag{2.42}$$

and

$$\mu(x,\theta,\alpha) = \int_{S^2} d\theta' M(x,\theta',\theta,\alpha). \tag{2.43}$$

Equation (2.40) becomes the equation we will later call (F):

$$\begin{aligned}\eta(x,\theta,\alpha) &= \eta(x,-\theta,-\alpha) + \mu(x,\theta,\alpha)\\ &+ \int_{-\infty}^{\infty} d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\, \eta(x,-\theta',\beta), \quad -\infty < \alpha < \infty.\end{aligned} \tag{2.44)=(F}$$

This equation is fully equivalent to the basic S-matrix relation, Equation (2.36). Indeed, if Equation (2.44) holds, then Fourier inverting it yields Equation (2.36) with the same A used in Equation (2.42) to define M. This has an important consequence for our work in Section 3: suppose we know that u is outgoing, that is

$$u(x,\theta,k) = e^{ik\theta\cdot x} + A_q(\theta',\theta,k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r \to \infty, \tag{2.45}$$

and that this $u(x,\theta,k)$ obeys (2.36) but with $A(\theta',\theta,k)$ in that equation *not* assumed equal to $A_q(\theta',\theta,k)$. Then, by using (2.45) in (2.36), taking $r \to \infty$ in (2.36) and comparing the asymptotic behavior of both sides, we find $A = A_q$. (This is the method we used in formulas (2.25)–(2.28).)

The procedures leading from (2.36) to (2.44) or from (2.44) back to (2.36) only require the possibility of taking Fourier transforms and inverting them. This should be understood in the sense of distributions. If $q \in Q$, then $A(\theta',\theta,k)$ decreases fast enough in k in some suitable operator norm and the kernels M are kernels of Fredholm operators on $L^2(\mathbf{R}_+ \times S^2)$ (see a discussion of this point in [2, p. 26]).

As it stands, Equation (2.44) is true for all α. The function η can be simplified for $\alpha < 0$. Recall that $\nu(x,\theta,k)$ is meromorphic for $k \in \mathbf{C}^+$ and vanishes as $k \to \infty$, so for $\alpha < 0$, we can evaluate the Fourier transform by contour integration:

$$\begin{aligned}\eta(x,\theta,\alpha) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik\alpha}\, \nu(x,\theta,k)\\ &= \oint \frac{dk}{2\pi}\, e^{-ik\alpha}\, \nu(x,\theta,k),\end{aligned} \tag{2.46}$$

where the contour is closed at $|k| \to \infty$ in $\mathbb{C}^+$. Thus, for $\alpha < 0$,

$$\eta(x,\theta,\alpha) = i \sum_{\lambda_j} e^{\lambda_j \alpha} \operatorname*{Res}_{k=i\lambda_j} \nu(x,\theta,k), \tag{2.47}$$

where the sum is over the simple poles of $\nu(x,\theta,k)$ at $k = i\lambda_j, \lambda_j > 0$. These poles correspond to the bound states generated by the potential $q(x)$. If there are no bound states, which we assume throughout this paper for simplicity, then

$$\eta(x,\theta,\alpha) = 0, \quad \alpha < 0. \tag{2.48}$$

The presence of bound states causes technical complications in the subsequent development, and the essential parts of the argument are easier to understand without these complications. This is why we consider only *potentials that do not produce bound states.* For these potentials, Equation (2.44) is the *pair* of equations (with (2.48) assumed)

$$\eta(x,\theta,\alpha) = \mu(x,\theta,\alpha) + \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\, \eta(x,-\theta',\beta), \alpha > 0, \tag{2.49)=(M}$$

and

$$\eta(x,-\theta,-\alpha) = -\mu(x,\theta,\alpha) - \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\, \eta(x,-\theta',\beta), \alpha < 0. \tag{2.50}$$

It is convenient to rewrite (2.50), putting $\alpha \to -\alpha$ and $\theta \to -\theta$. This gives

$$\eta(x,\theta,\alpha) = -\mu(x,-\theta,-\alpha) - \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',-\theta,-\alpha+\beta)\eta(x,-\theta',\beta), \alpha > 0. \tag{2.51) = (M'}$$

The Equations (2.49) and (2.51) are the starting point of an approach to inverse scattering pioneered by Newton. We will call them (M) and (M'), respectively. (M) is an equation of Marchenko type: to find $\eta(\alpha)$ only values of the kernel for arguments greater than α are needed. It is worth stressing that if we have a function η that solves (2.49) and (2.51) only for $\alpha > 0$, then we can find a function $\nu(x,\theta,k)$ this is analytic for k in $\mathbb{C}^+$ and solves (2.40). The Fourier transform of this ν, then, is equal to η for $\alpha > 0$ and vanishes for $\alpha < 0$. Thus, (2.49) and (2.51) alone, without assumptions about η for $\alpha < 0$, are enough to recover (2.40) and then the basic S-matrix relation Equation (2.36). To see all this explicitly, let $\eta(x,\theta,\alpha)$ solve (2.49) and (2.51) for $\alpha > 0$. Define

$$\nu(x,\theta,k) = \int_0^\infty d\alpha\, e^{ik\alpha}\, \eta(x,\theta,\alpha). \tag{2.52}$$

This ν is analytic for k in $\mathbb{C}^+$ if η is square integrable in α. Then, multiplying (2.49) by $e^{ik\alpha}$ and integrating in α from 0 to ∞, we get

$$\nu(x,\theta,k) = \int_0^\infty \mu(x,\theta,\alpha)\, e^{ik\alpha} d\alpha + \int_0^\infty d\alpha \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\, e^{ik\alpha}\, \eta(x,-\theta',\beta). \tag{2.53}$$

Now extend the limits of integration in α to $-\infty$ and subtract the terms this adds. One gets

$$\begin{aligned} \nu(x,\theta,k) = &\int_{-\infty}^\infty \mu(x,\theta,\alpha)\, e^{ik\alpha}\, d\alpha + \int_{-\infty}^\infty d\alpha \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\, e^{ik\alpha}\, \eta(x,-\theta',\beta) \\ &- \int_{-\infty}^0 \mu(x,\theta,\alpha)\, e^{ik\alpha}\, d\alpha - \int_{-\infty}^0 d\alpha \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\, e^{ik\alpha}\, \eta(x,-\theta',\beta). \end{aligned} \tag{2.54}$$

Now change $\alpha \to -\alpha$ in the last two integrals and use the definitions of μ and M in the first two

$$\begin{aligned}\nu(x,\theta,k) &= \int_{S^2} d\theta' \frac{ik}{2\pi} A(\theta',\theta,k)\, e^{ik(\theta'-\theta)\cdot x} \\ &+ \int_{S^2} d\theta' \frac{ik}{2\pi} A(\theta',\theta,k)\, e^{ik(\theta'-\theta)\cdot x} \int_0^\infty d\beta\, e^{-ik\beta}\, \eta(x,-\theta',\beta) \\ &+ \int_0^\infty d\alpha\, e^{-ik\alpha} \left[-\mu(x,\theta,-\alpha) - \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,-\alpha+\beta)\, \eta(x,-\theta',\beta) \right].\end{aligned} \tag{2.55}$$

Thus, if η solves (2.51) for $\alpha > 0$, we have

$$\begin{aligned}\nu(x,\theta,k) &= \int_{S^2} d\theta' \frac{ik}{2\pi} A(\theta',\theta,k)\, e^{ik(\theta'-\theta)\cdot x} \\ &+ \int_{S^2} d\theta' \frac{ik}{2\pi} A(\theta',\theta,k)\, e^{ik(\theta'-\theta)\cdot x}\, \nu(x,-\theta',-k) + \nu(x,-\theta,-k),\end{aligned} \tag{2.56}$$

which is exactly (2.40). This result means that (2.49) and (2.51) for $\alpha > 0$ already contain the information that $q(x)$ does not produce bound states, which is the reason η vanishes for $\alpha < 0$.

Do Equations (2.49) and (2.51) have a solution? If the function A that defines the kernel M in Equation (2.42) is in fact the scattering ampltitude A_q corresponding to a potential $q \in Q$, then we know that (2.49) and (2.51) *do* have the solution η (which we here call η_q to emphasize the dependence on q) defined by Equations (2.37), (2.39) and (2.41). Furthermore, this solution η_q can be used to determine the potential q. One obvious way is to undo Equations (2.41) and (2.39) to obtain u_q from η_q. Then

$$(\nabla^2 + k^2)\, u_q(x,\theta,k) = q(x)\, u_q(x,\theta,k), \tag{2.57}$$

and so (for θ, k such that $u(x,\theta,k) \neq 0$)

$$\frac{[(\nabla^2 + k^2)\, u_q(x,\theta,k)]}{u_q(x,\theta,k)} = q(x). \tag{2.58}$$

We could also obtain $q(x)$ directly from $\eta_q(x,\theta,\alpha)$ in the following way.
Compute $\nabla^2\eta_q$ and $2\theta\cdot\nabla\frac{\partial}{\partial\alpha}\eta_q$ using (2.41) and (2.39):

$$\begin{aligned}\nabla^2\eta_q &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik\alpha} e^{-ik\theta\cdot x} [-k^2 u_q(x,\theta,k) - 2ik\theta\cdot\nabla u_q(x,\theta,k) \\ &+ \nabla^2 u_q(x,\theta,k)],\end{aligned} \tag{2.59}$$

$$2\theta\cdot\nabla\frac{\partial}{\partial\alpha}\eta_q = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik\alpha}\, e^{-ik\theta\cdot x}(-ik)[-2\theta\cdot ik\theta\, u_q(x,\theta,k) + 2\theta\cdot\nabla u_q(x,\theta,k)]. \tag{2.60}$$

Thus,

$$\begin{aligned}\left(\nabla^2 - 2\theta\cdot\nabla\frac{\partial}{\partial\alpha}\right)\eta_q(x,\theta,\alpha) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik\alpha} e^{-ik\theta\cdot x} (\nabla^2 + k^2)\, u_q(x,\theta,k) \\ &= q(x) \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik\alpha} [e^{-ik\theta\cdot x} u_q(x,\theta,k) - 1 + 1].\end{aligned} \tag{2.61}$$

So, we find

$$\left(\nabla^2 - 2\theta\cdot\nabla\frac{\partial}{\partial\alpha}\right)\eta_q(x,\theta,\alpha) = q(x)\,\eta_q(x,\theta,\alpha) + q(x)\,\delta(\alpha). \tag{2.62}$$

Now integrate (2.62) in α from $-\epsilon$ to ϵ, recall that $\eta_q(x,\theta,\alpha) = 0$ for $\alpha < 0$, and then take the limit $\epsilon \to 0$. This gives

$$-2\theta\cdot\nabla\,\eta_q(x,\theta,\alpha = 0^+) = q(x), \tag{2.63}$$

and so, η_q directly determines $q(x)$.

Equations (2.63) and (2.58) are compatibility conditions on η_q, which are necessary for $A(\theta',\theta,k)$ to be the scattering amplitude corresponding to a local potential. Condition (2.63) is called miraculous in [2] because the left-hand side of (2.63) appears to depend on θ as well as x, while the right-hand side is a function of x only. Condition (2.58), introduced in [3] and used in [4] is also miraculous in this sense. The left-hand side of (2.58) appears to depend on θ, k and x but for $\eta = \eta_q$ the θ and k dependence miraculously disappear. In Section 3, we will refer to Equation (2.63) as the miracle compatibility condition, or briefly, the miracle condition. (Our derivation of (2.63) differs from the derivation in [2].)

We have seen that on the assumption $A = A_q$, $q \in Q$, then (2.49) and (2.51) have a solution $\eta = \eta_q$. In the next section, we will see to what extent this procedure can be inverted, a potential q found, and the equality $A = A_q$ proved. The critical assumption that makes recovery of $q(x)$, the potential in the Schrödinger equation, possible is that equation (M) must have a *unique* solution. Newton [5] has recently published a proof of this if A is in fact A_q for some potential q in a specified class. His proof relies on the uniqueness of the solution to a certain Goursat-type problem, which uniqueness he asserts in [5, p. 2423]. We have found a counter example to this latter assertion (see Section 4), and so the status of the assumption that (M) has a unique solution is unclear to us. A uniqueness theorem for a Gousat-type problem is proved in [9].

3. THE INVERSE PROBLEM

This section deals with the characterization and inversion problems introduced in Section 2.

3.1. The Marchenko Equation and its Consequences

We start with a function $A(\theta',\theta,k)$ *not* assumed *a priori* to be a scattering amplitude corresponding to a potential in a particular class and construct, under some assumptions, a potential q and a function ψ that obeys the Schrödinger equation with this potential. Using $A(\theta',\theta,k)$, we construct the pair of Equations (2.49):=(M) and (2.51):=(M')

$$\eta(x,\theta,\alpha) = \mu(x,\theta,\alpha) + \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\,\eta(x,-\theta',\beta), \qquad \alpha > 0, \tag{3.1) = (M}$$

$$\eta(x,\theta,\alpha) = -\,\mu(x,-\theta,-\alpha) - \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,-\alpha+\beta)\,\eta(x,-\theta',\beta), \qquad \alpha > 0. \tag{3.2) = (M'}$$

The kernel M and inhomogeneity μ are defined in (2.42) and (2.43), respectively, using the given $A(\theta',\theta,k)$. We work with Equation (M), in this section, and return to (M'), in Section 3.3.

Suppose that η is a solution of (M). Define

$$\Gamma(x,\theta,\alpha) := \left(\nabla^2 - 2\theta\cdot\nabla\,\frac{\partial}{\partial\alpha}\right)\eta(x,\theta,\alpha). \tag{3.3}$$

(We have seen Γ before: it is the left-hand side of (2.62).) We will see that *if* the miracle condition (2.63) holds,

$$-2\theta\cdot\nabla\,\eta(x,\theta,\alpha = 0^+) \equiv q(x), \tag{3.4}$$

then Γ satisfies (M) with μ replaced by $q(x)\mu$. This in turn implies that *if* (M) has a ***unique*** solution for all $x \in \mathbf{R}^3$ and all $\theta \in S^2$, then $\Gamma = q(x)\eta$; that is, for $\alpha > 0$,

$$\left(\nabla^2 - 2\theta \cdot \nabla \frac{\partial}{\partial\alpha}\right) \eta(x,\theta,\alpha) = q(x)\, \eta(x,\theta,\alpha), \quad \alpha > 0. \tag{3.5}$$

From (3.5), it is a simple matter to show that the function

$$\Psi_\eta(x,\theta,k) = e^{ik\theta\cdot x}\left[1 + \int_0^\infty d\alpha\, e^{ik\alpha}\,\eta(x,\theta,\alpha)\right], \tag{3.6}$$

satisfies the Schrödinger equation with the potential $q(x)$. Now we establish these assertions.

Apply $(\nabla^2 - 2\theta \cdot \nabla\frac{\partial}{\partial\alpha})$ to both sides of (3.1) (for $\alpha > 0$):

$$\begin{aligned}\left(\nabla^2 - 2\theta \cdot \nabla \frac{\partial}{\partial\alpha}\right) \eta(x,\theta,\alpha) &= \left(\nabla^2 - 2\theta \cdot \nabla \frac{\partial}{\partial\alpha}\right) \mu(x,\theta,\alpha) \\ &\quad + \int_0^\infty d\beta \int_{S^2} d\theta' \left(\nabla^2 - 2\theta \cdot \nabla \frac{\partial}{\partial\alpha}\right) M(x,\theta',\theta,\alpha+\beta) \\ &\quad \times\ \eta(x,-\theta',\beta), \qquad \alpha > 0.\end{aligned} \tag{3.7}$$

Now notice the following identities:

$$\nabla M(x,\theta',\theta,\alpha) = (\theta - \theta')\frac{\partial}{\partial\alpha} M(x,\theta',\theta,\alpha), \tag{3.8}$$

and so

$$\left(\nabla^2 - 2\theta \cdot \nabla \frac{\partial}{\partial\alpha}\right) M(x,\theta',\theta,\alpha) = 0, \tag{3.9}$$

which implies

$$\begin{aligned}&\left(\nabla^2 - 2\theta \cdot \nabla \frac{\partial}{\partial\alpha}\right) M(x,\theta',\theta,\alpha+\beta)\, \eta(x,-\theta',\beta) \\ &\qquad = -2\theta' \frac{\partial M(x,\theta',\theta,\alpha+\beta)}{\partial\alpha} \cdot \nabla\, \eta(x,-\theta',\beta) \\ &\qquad\quad + M(x,\theta',\theta,\alpha+\beta)\nabla^2\eta.\end{aligned} \tag{3.10}$$

Equation (3.10) implies that

$$\left(\nabla^2 - 2\theta \cdot \nabla \frac{\partial}{\partial\alpha}\right) \mu(x,\theta,\alpha) = 0, \tag{3.11}$$

and, using (3.11) and (3.10) in (3.7), we get (for $\alpha > 0$)

$$\begin{aligned}\left(\nabla^2 - 2\theta \cdot \nabla \frac{\partial}{\partial\alpha}\right) \eta(x,\theta,\alpha) &= \int_0^\infty d\beta \int_{S^2} d\theta' [-2\theta' \frac{\partial}{\partial\beta} M(x,\theta',\theta,\alpha+\beta) \cdot \nabla\, \eta(x,-\theta',\beta) \\ &\quad + M(x,\theta',\theta,\alpha+\beta)\nabla^2\eta(x,-\theta',\beta), \quad \alpha > 0.\end{aligned} \tag{3.12}$$

Now integrate the right-hand side by parts with respect to β, and use the definition of Γ (3.3) on the left-hand side:

$$\begin{aligned}\Gamma(x,\theta,\alpha) &= \int_{S^2} d^2\theta' (-2\theta' \cdot \nabla\eta(x,-\theta',\beta) M(x,\theta',\theta,\alpha+\beta)\,|_{\beta=0}^{\beta=\infty} \\ &\quad + \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\left[\nabla^2 + 2\theta' \cdot \nabla \frac{\partial}{\partial\beta}\right] \eta(x,-\theta',\beta), \quad \alpha > 0.\end{aligned} \tag{3.13}$$

If the $\beta = \infty$ limit vanishes and the miracle condition (3.4) holds, then (3.13) becomes

$$\Gamma(x,\theta,\alpha) = q(x)\mu(x,\theta,\alpha) + \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\,\Gamma(x,-\theta',\beta), \quad \alpha > 0. \tag{3.14}$$

Equation (3.14) is exactly (3.1) with μ replaced by $q(x)\mu$. Now, *if* the solution to (3.1) is ***unique***, then

$$\Gamma(x,\theta,\alpha) = q(x)\,\eta(x,\theta,\alpha), \tag{3.15}$$

and, recalling the definition of Γ in (3.3), we have

$$\left(\nabla^2 - 2\theta\cdot\nabla\frac{\partial}{\partial\alpha}\right)\eta(x,\theta,\alpha) = q(x)\,\eta(x,\theta,\alpha), \tag{3.16}$$

and we have derived (3.5) as promised. The assumptions we have used are the following ones, which we call:

ASSUMPTIONS (M).

(a) Equation (3.1) has a solution which is in $L^2(\mathbf{R}_+ \times S^2)$ and such that $|\eta| + |\nabla\eta| \to 0$ as $\alpha \to +\infty$,
(b) this solution is unique,
(c) this solution obeys the miracle compatibility condition (3.4) and generates $q \in Q$, where Q has been defined in the Introduction;
(d) the following limits hold: $M(x,\theta',\theta,\alpha) \to 0$ as $\alpha \to +\infty$, for any fixed $x \in \mathbf{R}^3$ and $|\theta' - \theta| \geq \delta > 0$; $M(x,\theta',\theta,\alpha) \to 0$ as $|x| \to \infty$, for any fixed $\alpha > 0$ and $|\theta' - \theta| \geq \delta > 0$ (δ is an arbitrary small fixed number).

Part (b) of assumptions (M) is equivalent to

(b$'$). The equation

$$y(x,\theta,\alpha) = \int_0^\infty d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\,y(x,-\theta',\beta) \tag{3.17}$$

has only the trivial solution. (We also remark that the need for the decay of M as $|x| \to \infty$, is not needed above but *will* be needed in Section 3.2.)

Now we show that the solution η guaranteed by assumptions (M) defines, via (3.6), a solution to the Schrödinger equation. Multiply both sides of (3.5) by $e^{ik\alpha}$ and integrate from 0 to ∞:

$$\int_0^\infty e^{ik\alpha}\left[\nabla^2 - 2\theta\cdot\nabla\frac{\partial}{\partial\alpha}\right]\eta(x,\theta,\alpha)\,d\alpha = \int_0^\infty q(x)\,e^{ik\alpha}\eta(x,\theta,\alpha)\,d\alpha. \tag{3.18}$$

Now integrate by parts on the left-hand side to get

$$\nabla^2\int_0^\infty e^{ik\alpha}\,\eta(x,\theta,\alpha)\,d\alpha - 2\theta\cdot\nabla\,\eta(x,\theta,\alpha)\,e^{ik\alpha}\big|_{\alpha=0}^{\alpha=\infty}$$
$$+\int_0^\infty d\alpha\,e^{ik\alpha}\,2ik\theta\cdot\nabla\,\eta(x,\theta,\alpha) = q(x)\int_0^\infty e^{ik\alpha}\,\eta(x,\theta,\alpha)\,d\alpha. \tag{3.19}$$

The upper limit, $\alpha = \infty$, vanishes by the assumption on η and the lower limit, $\alpha = 0$, gives $-q(x)$ by the miracle condition. Thus (3.19) becomes

$$(\nabla^2 + 2ik\theta\cdot\nabla)\int_0^\infty d\alpha\,e^{ik\alpha}\,\eta(x,\theta,\alpha) - q(x) = q(x)\int_0^\infty e^{ik\alpha}\,\eta(x,\theta,\alpha)\,d\alpha, \tag{3.20}$$

and with a little rearrangement

$$(\nabla^2 + 2ik\theta\cdot\nabla)\left[1+\int_0^\infty e^{ik\alpha}\eta(x,\theta,\alpha)\,d\alpha\right] = q(x)\left[1+\int_0^\infty e^{ik\alpha}\,\eta(x,\theta,\alpha)\,d\alpha\right]. \tag{3.21}$$

Now introduce ψ_η defined in (3.6):

$$(\nabla^2 + 2ik\theta\cdot\nabla)[e^{-ik\theta\cdot x}\psi_\eta(x,\theta,k)] = q(x)[e^{-ik\theta\cdot x}\psi_\eta(x,\theta,k)]. \tag{3.22}$$

Carrying out the derivatives on the left-hand side we finally get

$$(\nabla^2 + k^2)\psi_\eta(x,\theta,k) = q(x)\psi_\eta(x,\theta,k), \tag{3.23}$$

the Schrödinger equation we sought.

It is important to note what we have and have not shown under assumption (M): from $A(\theta',\theta,k)$, we have found a potential $q(x)$ and a function $\psi_\eta(x,\theta,k)$ which satisfies the Schrödinger equation with this potential. We summarize this by the diagram

$$\begin{array}{c} (M): A \rightarrow \eta \rightarrow \psi_\eta. \\ \quad\downarrow \\ \quad q \end{array}$$

We do *not* yet know what the asymptotic behavior of ψ_η is, not even that it is outgoing. This is the topic we take up in Section 3.2.

In assumptions (M) one might consider replacing (a) by requiring the data to be such that equation (M) has the Fredholm property; then existence of the solution would follow from its assumed uniqueness. This would not simplify the theory because the crucial assumption is the compatibility condition which is not algorithmically verifiable in terms of A directly.

Is the potential $q(x)$ real-valued? We can prove that it is real-valued *if* the function $A(\theta',\theta,k)$ obeys the reality property (2.31). To see this, note that if (2.31) holds, then the kernel $M(x,\theta',\theta,\alpha)$ is real-valued (start with the definition of $M(x,\theta',\theta,\alpha)$ in (2.42), then change the integration variable from $k\rightarrow -k$, then use (2.31)):

$$\begin{aligned} M(x,\theta',\theta,\alpha) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{ik}{2\pi}\,A(\theta',\theta,k)\,e^{-ik\alpha+ik(\theta'-\theta)\cdot x} \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left(\frac{-ik}{2\pi}\right)A(\theta',\theta,-k)\,e^{+ik\alpha-ik(\theta'-\theta)\cdot x} \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{\overline{ik}}{2\pi}\,\overline{A(\theta',\theta,k)}\quad\overline{e^{-ik\alpha+ik(\theta'-\theta)\cdot x}} \\ &= \overline{M(x,\theta',\theta,\alpha)}. \end{aligned} \tag{3.24}$$

This means that if η is a solution to (M), then so is $\bar\eta$. But one of the assumptions in (M) says that equation (M) has at most one solution. This implies that $\eta=\bar\eta$. This in turn means that $q(x)$ defined through the miracle condition (3.5) is real-valued. Notice that finding $q(x)$ by the above scheme and even finding that it is real-valued uses only assumptions (M) and the reality condition on A. Thus, it may be perfectly possible to start with, say, a non-unitary A, obtain a potential $q(x)$ and find the scattering amplitude A_q corresponding to this q. Clearly $A\neq A_q$. The point of this remark is that assumptions (M) are powerful but are hard to check and it would be good to get even more out of these assumptions.

3.2. ψ_η is Outgoing

We show in this section (under the assumptions (M) only) that ψ_η, defined in Equation (3.6), in fact obeys the outgoing boundary conditions, and so it determines the scattering amplitude A_q.

We cannot prove that $A_q = A$ without further assumptions (additional to (M)). The method of proof is indirect and lengthy; a shorter and direct proof would be highly desirable.

Consider the Schrödinger equation

$$(\nabla^2 + k^2 - q(x))\, u_q^+(x,\theta,k) = 0, \tag{3.25}$$

together with the outgoing boundary condition Equations (2.2)–(2.3), so that u_q^+ is the scattering solution. Then, from Section 2, we know that the function

$$\eta_q(x,\theta,\alpha) = \int\limits_{\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik\alpha}[e^{-ik\theta\cdot x}\, u_q^+(x,\theta,k) - 1], \tag{3.26}$$

solves equation (M) whose kernel is determined by A_q, η is miraculous, and η_q vanishes for $\alpha < 0$. Thus, η_q satisfies the same hyperbolic system (3.16) and (3.4) that η does. We are going to prove that the solution to this system is unique in a suitable class of functions, so that $\eta_q = \eta$. This means that the ψ_η, defined by Equation (3.6), has the scattering amplitude A_q but it does not show that $A_q = A$. The further assumptions that we need to prove that $A_q = A$ are introduced in Section 3.3.

To show that $\eta_q = \eta$ we first simplify the system. The equations we start with are (3.16) and (3.4) obeyed by both η_q and η:

$$\left(\nabla^2 - 2\theta\cdot\nabla\frac{\partial}{\partial\alpha}\right)\eta(x,\theta,\alpha) = q(x)\,\eta(x,\theta,\alpha), \quad \alpha > 0, \tag{3.27}$$

and

$$-2\theta\cdot\nabla\eta(x,\theta,\alpha = 0^+) = q(x). \tag{3.28}$$

Define

$$\hat\eta(x,\theta,\alpha) = \eta(x,\theta,\alpha - \theta\cdot x), \quad \alpha > \theta\cdot x. \tag{3.29}$$

Then

$$\nabla\hat\eta(x,\theta,\alpha) = (\nabla - \theta\,\frac{\partial}{\partial\beta})\,\eta(x,\theta,\beta)\,|_{\beta=\alpha-\theta\cdot x}, \tag{3.30}$$

and so $\hat\eta$ obeys

$$\left(\nabla^2 - \frac{\partial^2}{\partial\alpha^2}\right)\hat\eta(x,\theta,\alpha) = q(x)\,\hat\eta(x,\theta,\alpha), \quad \alpha > 0\cdot x. \tag{3.31}$$

Using (3.29)–(3.31), one gets $-2\theta\cdot\nabla_x\,\hat\eta(x,\theta,\theta\cdot x) = -2\theta\cdot\nabla_x\,\eta(x,\theta,0) = q(x)$. Thus,

$$-2\theta\cdot\nabla\,\hat\eta(x,\theta,\theta\cdot x) = q(x). \tag{3.32}$$

If we substitute $x + t\theta$, $t\in\mathbf{R}$, in place of x in (2.32), we obtain

$$-2\theta\cdot\nabla\hat\eta(x+t\theta,\theta,\theta\cdot x + t) = q(x+t\theta), \tag{3.33}$$

and Equation (3.33) is

$$-2\,\frac{\partial}{\partial t}\,\hat\eta(x+t\theta,\theta,\theta\cdot x+t) = q(x+t\theta). \tag{3.34}$$

Integrate (3.34) in t from 0 to ∞ and use $\hat\eta(x+t\theta,\theta,\theta\cdot x+t)|_{t=+\infty} = 0$ to get

$$\hat\eta(x,\theta,\theta\cdot x) = \frac{1}{2}\int\limits_0^\infty q(x+t\theta)\,dt. \tag{3.35}$$

Now we show that Equation (3.31) with condition (3.35) has at most one solution in the class of $\mathbf{C}^2$ functions $\hat\eta(x,\theta,\alpha)$ of x and α, $\hat\eta\in L^2(\mathbf{R})$ in α. First, if there were two solutions, say

$\hat{\eta}_1$ and $\hat{\eta}_2$, then $\hat{\eta}_1$—$\hat{\eta}_2$ satisfies the same Equation (3.31) and the condition $\hat{\eta}(x, \theta, \theta \cdot x) = 0$. So it is sufficient to show that the equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial\alpha^2} - q(x)\right) f(x, \alpha) = 0, \quad \alpha \geq \theta \cdot x, \tag{3.36}$$

with the conditions:

$$f(x, \theta \cdot x) = 0, \tag{3.37}$$

and

$$\lim_{\substack{|x| \to \infty \\ \alpha \geq \theta \cdot x}} f(x, \alpha) = 0 \quad \text{for any fixed } \alpha, \tag{3.38}$$

has only the trivial solution in the class of twice differentiable functions $f(x, \alpha)$, $x \in \mathbf{R}^3$, $\alpha \geq \theta \cdot x$, $\theta \in S^2$ is arbitrary fixed. We do this in two steps: first prove it for $q(x) = 0$, then show that if $q \not\equiv 0$, $q \in Q_a$, the problem can be reduced to an equation which has only the trivial solution. Our argument is similar to an argument given by R. Newton in [6].

STEP 1. Let $q(x) \equiv 0$ and let $\theta = e_3$, where e_3 is the unit vector along the x_3-axis so that Equations (3.36)–(3.38) become

$$\left(\nabla^2 - \frac{\partial^2}{\partial\alpha^2}\right) f(x, \alpha) = 0, \quad \alpha \geq x_3, \tag{3.39}$$

$$f(x, \alpha = x_3) = 0, \tag{3.40}$$

$$\lim_{\substack{|x| \to \infty \\ \alpha \geq x_3}} f(x, \alpha) = 0. \tag{3.41}$$

Let $x_\perp = (x_1, x_2)$, $\xi = (\alpha + x_3)/2$, $\zeta = (\alpha - x_3)/2$ and set $f(x, \alpha) = f(x_1, x_2, \xi - \zeta, \xi + \zeta) := g(x_\perp, \xi, \zeta)$. Then, (3.39) and (3.40) become

$$(\nabla_\perp^2 - \partial_{\xi\zeta})\, g = 0, \quad \zeta \geq 0, \tag{3.42}$$

$$g(x_\perp, \xi, 0) = 0. \tag{3.43}$$

Fourier transform in $x_\perp$ and Laplace transform in ζ to obtain

$$(-k^2 - \lambda\partial_\xi)\, \tilde{g}(k, \xi, \lambda) = 0, \quad \text{Re } \lambda > 0, \tag{3.44}$$

where (3.43) is taken into account. The general solution of (3.44) is

$$\tilde{g}(k, \xi, \lambda) = C(\lambda, k)\, e^{-k^2\xi/\lambda}. \tag{3.45}$$

If $x_3 \to -\infty$, then $\zeta > 0$, $\xi \to -\infty$, and in this limit f and, hence, $\tilde{g}$ must vanish. So $C = 0$ and then $\tilde{g} = 0$. Thus, $f = 0$. An alternative proof is given after the lemmas and proofs below.

STEP 2. Now we reconsider (3.36)–(3.38) with $q \not\equiv 0$. Define $\tilde{f}(x, \alpha) = \theta(\alpha - x_3) f(x, \alpha)$:

$$\theta(\alpha) := \begin{cases} 1 & \alpha < 0, \\ 0 & \alpha \geq 0. \end{cases} \tag{3.46}$$

This function $\tilde{f}(x, \alpha)$ is *not* a global solution of the hyperbolic Equation (3.36) but *is* a solution of (3.36) for $\alpha > x_3$. Let G solve the equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial\alpha^2}\right) G(x, \alpha) = -q(x)\tilde{f}(x, \alpha), \quad \forall x \in \mathbf{R}^3, \quad \forall \alpha \in \mathbf{R}^1. \tag{3.47}$$

The function

$$G(x,\alpha) = \int_{\mathbf{R}^4} \frac{q(y)\,\tilde{f}(y,\alpha')\,\delta(\alpha' - \alpha + |x-y|)}{4\pi|x-y|}\,d^3y\,d\alpha' \tag{3.48}$$

solves (3.47). For $\alpha > x_3$ one has $\tilde{f} = f$, and Equations (3.36) and (3.47) yield

$$\left(\nabla^2 - \frac{\partial^2}{\partial\alpha^2}\right)(f+G) = 0, \quad \alpha > x_3. \tag{3.49}$$

For $\alpha = x_3$, the integrand in (3.48) contains $\theta(\alpha' - y_3) = \theta(x_3 - |x-y| - y_3)$. Thus, the region of integration is the set of points where $x_3 - y_3 - |x-y| \geq 0$. Since $x_3 - y_3 \geq |x-y|$ implies $x_3 \geq y_3$, $x_1 = y_1$, $x_2 = y_2$, one has $x_3 = y_3 + \gamma$, $\gamma \geq 0$, that is

$$y = x - \gamma e_3,\ \gamma \geq 0,\ y = y_3 e_3,\ x = x_3 e_3. \tag{3.50}$$

This means that $G(x,\alpha = x_3) = 0$, and since $f(x,\alpha = x_3) = 0$ by (3.40), one gets

$$(f+G)(x,\alpha = x_3) = 0. \tag{3.51}$$

But (3.49) and (3.51) are exactly the Equations (3.39) and (3.40) of Step 1, and (3.41) for the function $f+G$ follows from (3.48), so $f+G=0$ for all $x \in \mathbf{R}^3$ and $\alpha \geq x_3$ by the result obtained in Step 1. Thus, for $\alpha > x_3$, we have (this is proved in Lemma 2 below)

$$f(x,\alpha) = -\int_{\mathbf{R}^3} \frac{q(y) f(y,\alpha - |x-y|)\,\theta\,(\alpha - |x-y| - y_3)}{4\pi|x-y|}\,d^3y, \tag{3.52}$$

and for $\alpha < x_3$ the integral (3.52) vanishes. So we can multiply (3.52) by $\theta(\alpha - x_3)$ and get

$$\tilde{f}(x,\alpha) = -\int_{\mathbf{R}^3} \frac{q(y)\tilde{f}(y,\alpha - |x-y|)}{4\pi|x-y|}\,d^3y, \quad \forall\alpha \in \mathbf{R}^1. \tag{3.53}$$

Now take the Fourier transform in α $\left(\hat{\tilde{f}} := \int_{-\infty}^{\infty} \tilde{f}(x,\alpha)\,e^{i\omega\alpha}d\alpha\right)$

$$\hat{\tilde{f}}(x,\omega) = -\int_{\mathbf{R}^3} \frac{q(y)e^{i\omega|x-y|}}{4\pi|x-y|}\,\hat{\tilde{f}}(x,\omega)\,d^3y, \quad \forall\omega \in \mathbf{R}^1. \tag{3.54}$$

In the following Lemma 1, we prove that (3.54) implies $\hat{\tilde{f}} = 0$ for $\omega \neq 0$ if $q \in Q_a$. If $\hat{\tilde{f}}(x,\omega)$ is locally in L^2 in the ω-variable, one concludes that $\hat{\tilde{f}} = 0$ and so f in (3.39) vanishes. Therefore, Equation (3.27) with condition (3.28) has at most one solution. But we know that Equations (3.27) and (3.28) have the solutions η_q and η, so we have proved that $\eta = \eta_q$.

This conclusion, $\eta = \eta_q$, allows us to extend the diagram just below Equation (3.23):

$$\begin{array}{llll} (M): & A \rightarrow & \eta \rightarrow \psi_\eta & \\ & & \downarrow \quad\ \ \downarrow & \psi_\eta = u_q. \\ & & q \rightarrow u_q & \end{array}$$

The asymptotic behavior of u_q determines the scattering amplitude A_q and so we may write

$$\begin{array}{ll} (M): & A \rightarrow \eta \rightarrow \psi_\eta \rightarrow A_q \\ & \qquad\ \ \downarrow \quad\ \downarrow \\ & \qquad\ \ q \rightarrow u_q \rightarrow A_q \end{array},$$

or, briefly,

$$(M): \quad A \rightarrow q \rightarrow A_q.$$

We will need further assumptions to prove that $A_q = A$.

REMARK. In Lemma 2.4.6 of [2], the crucial further assumption (assumptions (M') below (3.77) in this paper) is missing. Without it, one can only prove $\psi_\eta = u_q$ as we did above, but not that $A_q = A$. In Theorem 2.4.7 of [2], the assumption is made that an operator G^* defined there does not have the eigenvalue -1. This assumption implies (M') and so allows one to close the loop as we do in Section 3.3.

We now pass to the proofs of the two lemmas mentioned above.

LEMMA 1. *Assume that $q \in Q_a$. Then Equation (3.54) for all $\omega \in \mathbf{R}^1/\{0\}$ has only the trivial solution with the assumptions (2.22) in $\mathbf{C}(\mathbf{R}^3)$ and in $L^2(\mathbf{R}^3, \beta)$, $\beta > 1$.*

By $\mathbf{C}(\mathbf{R}^3)$ the space of continuous functions in $\mathbf{R}^3$ with the usual sup norm is denoted, $L^2(\mathbf{R}^3, \beta)$ is the space of $L^2_{\mathrm{loc}}(\mathbf{R}^3)$ functions with the norm $\|f\|_\beta := \{\int_{\mathbf{R}^3} |f|^2(1 + |x|^{-\beta})dx\}^{1/2}$.

PROOF OF LEMMA 1. [10] Let us assume that $\hat{\hat{f}} := h(x, \omega)$ belongs to $\mathbf{C}(\mathbf{R}^3)$, $\omega \in \mathbf{R}^1 \setminus 0$ is fixed. It follows from (3.53) that if $h \in L^2(\mathbf{R}^3, \beta)$, $\beta > 1$, then $h \in \mathbf{C}(\mathbf{R}^3)$, therefore, it is sufficient to consider Equation (3.53) in $\mathbf{C}(\mathbf{R}^3)$. Note that, if $q \in Q_a$ (and also if $q \in Q(\beta)$, $\beta > 2$), one has $h = o(|x|^{-1})$ for $|x| \to \infty$. The function h solves the equation

$$\ell_q h := [\nabla^2 + \omega^2 - q(x)]\, h = 0 \quad \text{in} \quad \mathbf{R}^3, \tag{3.55}$$

and satisfies the radiation condition

$$\lim_{R\to\infty} \int_{|x|=R} |h_r - i\omega h|^2\, ds = 0. \tag{3.56}$$

It follows from (3.55) that

$$\lim_{R\to\infty} \int_{|x|=R} [|h_r|^2 + \omega^2 |h|^2]\, ds = 0. \tag{3.57}$$

Indeed, since $q(x) = \overline{q(x)}$, Green's formula applied to the identity $\bar{h}\ell_q h - h\ell_q \bar{h} = 0$ yields

$$\lim_{R\to\infty} \int_{|x|=R} [h(s)\, \overline{h_r(s)} - \bar{h}(s)\, h_r(s)]\, ds = 0. \tag{3.58}$$

On the other hand,

$$\int_{|x|=R} |h_r - i\omega h|^2\, ds = \int_{|x|=R} [|h_r|^2 + \omega^2|h|^2]\, ds + i\omega \int_{|x|=R} [\bar{h}(s)\, h_r(s) - \overline{h_r}(s)\, h(s)]\, ds. \tag{3.59}$$

From (3.56), (3.58) and (3.59), Equation (3.57) follows. If $q \in Q_a$, then any solution to (3.54) solves

$$(\nabla^2 + \omega^2)\, h = 0 \quad \text{if} \quad |x| > a. \tag{3.60}$$

Multiply (3.60) by $Y_\ell(\alpha)$, $\alpha = xr^{-1}$, $r = |x|$, where $Y_\ell(\alpha) = Y_{\ell m}(\alpha)$, $-\ell \le m \le \ell$, are the orthonormal in $L^2(S^2)$ spherical harmonics, and integrate over S^2 to get:

$$h_\ell'' + \frac{2}{r} h_\ell' + \left[\omega^2 - \frac{\ell(\ell+1)}{r^2}\right] h_\ell = 0,\ r > a,\ h_\ell := (h, Y_\ell(\alpha))_{L^2(S^2)}. \tag{3.61}$$

The general solution to (3.61) is $h_\ell = (Z_{\ell+\frac{1}{2}}(\omega r)/(r^{1/2}))$, where $Z_\ell(r)$ is the general cylindrical function. One has

$$h_\ell \sim \frac{C_{1\ell} \cos\left[\omega r - (\ell+1)\frac{\pi}{2}\right] + C_{2\ell} \sin\left[\omega r - (\ell+1)\frac{\pi}{2}\right]}{r}, \quad r \to \infty, \tag{3.62}$$

where $C_{j\ell} = \text{const}$, $j = 1, 2$. From (3.62) and (3.57), it follows that $C_{1\ell} = C_{2\ell} = 0$ and $h_\ell = 0$, for all $\ell = 0, 1, 2 \dots$ Therefore, $h(x) = 0$ for $|x| > a$. Since $h(x)$ solves the elliptic equation (3.55) and $q \in Q_a$, one concludes by the unique continuation result [11, p. 14] that $h(x, \omega) \equiv 0$ in $\mathbf{R}^3$ for all real $\omega \neq 0$. Lemma 1 is proved. ∎

LEMMA 2. *The function $f(x, \alpha) + G(x, \alpha)$, $\alpha > x_3$, $\theta = e_3$, satisfies (3.41).*

PROOF. Since $f(x, \alpha)$ satisfies (3.41) it is sufficient to check that G satisfies (3.41). Write (3.48) as

$$G(x, \alpha) = \int_{\mathbf{R}^3} dy \frac{q(y) f(y, \alpha - |x - y|)\, \theta(\alpha - |x - y| - y_3)}{4\pi |x - y|}. \tag{3.63}$$

If $\alpha \geq x_3$ is fixed, it follows from (3.63), for $q \in Q_a$, that $|G(x, \alpha)| = o(|x|^{-1})$ as $x \to +\infty$, so that $G(x, \alpha) \to 0$ as $|x| \to \infty$, for any fixed $\alpha \geq x_3$. Lemma 2 is proved. ∎

REMARK. An alternative proof of the uniqueness result obtained below Equation (3.39) is as follows. The general solution to (3.39) is

$$f(x, \alpha) = \int_{\mathbf{R}^3} d\lambda\, e^{i\lambda \cdot x} [a(\lambda)\, e^{i|\lambda|\alpha} + b(\lambda)\, e^{-i|\lambda|\alpha}].$$

Condition (3.40) implies that

$$0 = \int_{\mathbf{R}^3} d\lambda\, e^{i\lambda \cdot x} [a(\lambda)\, e^{i|\lambda| x_3} + b(\lambda)\, e^{-i|\lambda| x_3}]$$

where we have taken the $\hat{e}_3$ axis to lie along θ. Taking the Fourier transform in x gives (with $\mu = (\mu_1, \mu_2, \mu_3)$ the Fourier transform parameter)

$$\begin{aligned} 0 = \int_{-\infty}^{\infty} d\lambda_3 [& a(\mu_1, \mu_2, \lambda_3)\, \delta(\lambda_3 + \sqrt{\mu_1^2 + \mu_2^2 + \lambda_3^2} - \mu_3) \\ & + b(\mu_1, \mu_2, \lambda_3)\, \delta(\lambda_3 - \sqrt{\mu_1^2 + \mu_2^2 + \lambda_3^2} - \mu_3)]. \end{aligned} \tag{3.64}$$

It follows from Equation (3.64) that for all μ_1, μ_2 and for $\mu_3 > 0$,

$$a\left(\mu_1, \mu_2, \frac{\mu_3^2 - \mu_1^2 - \mu_2^2}{2\mu_3}\right) = 0,$$

and this implies that $a(\mu) = 0$, for all μ. Similarly, for $\mu_3 < 0$ and all μ_1, μ_2, Equation (3.64) implies that

$$b\left(\mu_1, \mu_2, \frac{\mu_3^2 - \mu_1^2 - \mu_2^2}{2\mu_3}\right) = 0,$$

and this means that $b(\mu) = 0$, for all μ. Returning to the general solution to (3.39), we see that $f(x, \alpha)$ must vanish for all (x, α).

3.3 Closing the Loop

Starting with a function $A(\theta', \theta, k)$ and using assumptions (M), we have found a potential $q(x)$ and a function $\psi_\eta(x, \theta, k)$ which equals $u_q(x, \theta, k)$, the scattering solution to the Schrödinger equation with potential $q(x)$. Thus the asymptotic behavior of $\psi_\eta = u_q$ is

$$\psi_\eta(x, \theta, k) = e^{ik\theta \cdot x} + A_q(\theta', \theta, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r \to \infty,\ \theta' = x/r, \tag{3.65}$$

with $A_q(\theta', \theta, k)$ the scattering amplitude corresponding to the constructed potential $q(x)$. We have described this by the diagram

$$A \to q \to A_q. \tag{3.66}$$

Now we want to "close the loop" and prove that $A_q = A$.

An obvious way is to check and see if the constructed A_q does in fact equal A. This way, however, does not rely directly on the algorithmically verifiable properties of $A(\theta', \theta, k)$. An equivalent method is to insert η, the miraculous solution to (M) into (M'), Equation (2.51), and see if it is a solution to this equation. If it is, then η is a solution to both (M)=(2.49) and (M')=(2.51) and, hence, is a solution to the full Equation (F)=(2.44). As emphasized below Equation (2.44), this means that ψ_η defined from η by (3.6) has the asymptotic behavior

$$\psi_\eta(x, \theta, k) = e^{ik\theta\cdot x} + A(\theta', \theta, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \; r \to \infty, \; \theta' = x/r. \tag{3.67}$$

Thus, $A = A_q$ and the loop is closed.

Both of these methods are algorithmically verifiable procedures to check that $A = A_q$ and, in this sense, they provide reasonable ways to close the loop and solve the inverse problem. They are not quite satisfactory because neither one relies directly on properties of A, but requires calculations involving the solution to Equation (M). No conditions are known which are both directly verifiable in terms of the data A and also sufficient to close the loop. The conditions we will find—assumptions (M') below—share this drawback, but do show the role played by the classical conditions on A.

We begin by iterating equation (M)

$$\begin{aligned}\eta(x, \theta, \alpha) = {} & \mu(x, \theta, \alpha) + \int_0^\infty d\beta \int_{S^2} d\theta' M(x, \theta', \theta, \alpha+\beta)\, \mu(x, -\theta', \beta) \\ & + \int_0^\infty d\beta \int_{S^2} d\theta' M(x, \theta', \theta, \alpha+\beta) \int_0^\infty d\gamma \int_{S^2} d\theta'' M(x, \theta'', -\theta', \beta+\gamma) \\ & \times \; \eta(x, -\theta'', \gamma) \qquad \alpha > 0.\end{aligned} \tag{3.68}$$

Now use the definition of μ in (2.43), extend the limits of the β integration to $-\infty$, and subtract the terms this adds. We assume that orders of integration can be freely interchanged. Equation (3.68) becomes, for $\alpha > 0$,

$$\begin{aligned}\eta(x, \theta, \alpha) = {} & \mu(x, \theta, \alpha) + \int_{S^2} d\theta'' \left[\int_{-\infty}^\infty d\beta \int_{S^2} d\theta' M(x, \theta', \theta, \alpha+\beta)\, M(x, \theta'', -\theta', \beta)\right] \\ & + \int_0^\infty d\gamma \int_{S^2} d\theta'' \left[\int_{-\infty}^\infty d\beta \int_{S^2} d\theta' M(x, \theta', \theta, \alpha+\beta)\, M(x, \theta'', -\theta', \beta+\gamma)\right] \\ & \times \; \eta(x, -\theta'', \gamma) - \int_{-\infty}^0 d\beta \int_{S^2} d\theta'' \int_{S^2} d\theta' M(x, \theta', \theta, \alpha+\beta)\, M(x, \theta'', -\theta', \beta) \\ & - \int_{-\infty}^0 d\beta \int_{S^2} d\theta'' \int_0^\infty d\gamma \int_{S^2} d\theta' M(x, \theta', \theta, \alpha+\beta)\, M(x, \theta'', -\theta', \beta+\gamma) \\ & \times \; \eta(x, -\theta'', \gamma) \qquad \alpha > 0.\end{aligned} \tag{3.69}$$

To proceed, we will need some formulas involving $M(x, \theta', \theta, \alpha)$, Equations (3.70)–(3.72) below. They are obtained from the classical conditions on A.

We have seen earlier in (3.24) that $M(x,\theta',\theta,\alpha)$ is real-valued if $A(\theta',\theta,k)$ obeys the reality property

$$M(x,\theta',\theta,\alpha)=\overline{M(x,\theta',\theta,\alpha)}. \tag{3.70}$$

It is trivial to verify that, if $A(\theta',\theta,k)$ satisfies reciprocity, then so does M:

$$M(x,\theta',\theta,\alpha)=M(x,-\theta,-\theta',\alpha). \tag{3.71}$$

Finally, the unitarity property for $A(\theta',\theta,k)$, in addition to reality and reciprocity, leads to the following identity on $M(x,\theta',\theta,\alpha)$:

$$\begin{aligned} I(x,\theta'',\theta,\alpha,\gamma) :&= \int_{-\infty}^{\infty} d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\, M(x,\theta'',-\theta',\beta+\gamma) \\ &= -M(x,\theta'',-\theta,\gamma-\alpha)-M(x,-\theta'',\theta,\alpha-\gamma). \end{aligned} \tag{3.72}$$

We prove (3.72) at the end of this section (see Equations (3.78)–(3.82)). Returning to Equation (3.69) and, using the definition of $I(x,\theta',\theta,\alpha,\gamma)$ from (3.72), we have

$$\begin{aligned} \eta(x,\theta,\alpha) &= \mu(x,\theta,\alpha)+\int_{S^2} d\theta'' I(x,\theta'',\theta,\alpha,0) \\ &+\int_0^{\infty} d\gamma \int_{S^2} d\theta'' I(x,\theta'',\theta,\alpha,\gamma)\,\eta(x,-\theta'',\gamma) \\ &-\int_{-\infty}^{0} d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\,\mu(x,-\theta',\beta) \\ &-\int_{-\infty}^{0} d\beta \int_{S^2} d\theta' \int_0^{\infty} d\gamma \int_{S^2} d\theta'' M(x,\theta',\theta,\alpha+\beta)\, M(x,\theta'',-\theta',\beta+\gamma) \\ &\times\ \eta(x,-\theta'',\gamma) \qquad \alpha>0. \end{aligned} \tag{3.73}$$

Using (3.72), the definition of μ, and setting $\beta\to-\beta$ in the last two integrals gives

$$\begin{aligned} \eta(x,\theta,\alpha) &= \mu(x,\theta,\alpha)-\mu(x,\theta,\alpha)-\mu(x,-\theta,-\alpha) \\ &-\int_0^{\infty} d\gamma \int_{S^2} d\theta'' M(x,\theta'',-\theta,\gamma-\alpha)\,\eta(x,-\theta'',\gamma) \\ &-\int_0^{\infty} d\gamma \int_{S^2} d\theta'' M(x,-\theta'',\theta,\alpha-\gamma)\,\eta(x,-\theta'',\gamma) \\ &-\int_0^{\infty} d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha-\beta)\,\mu(x,-\theta',-\beta) \\ &-\int_0^{\infty} d\beta \int_{S^2} d\theta' M(x,\theta',\theta,\alpha-\beta)\,\mu(x,-\theta',-\beta) \\ &-\int_0^{\infty} d\beta \int_{S^2} d\theta' \int_0^{\infty} d\gamma \int_{S^2} d\theta'' M(x,\theta',\theta,\alpha-\beta)\, M(x,\theta'',-\theta',\gamma-\beta) \\ &\times\ \eta(x,-\theta'',\gamma) \qquad \alpha>0. \end{aligned} \tag{3.74}$$

The first and second terms cancel. Bring the third and fourth to the left hand side, and change the dummy variables γ and θ'' in the fifth term to β and θ'. This gives

$$\begin{aligned}\eta(x,\theta,\alpha)+\mu(x,-\theta,-\alpha)+\int_0^\infty d\gamma\int_{S^2} d\theta'' M(x,\theta'',-\theta,\gamma-\alpha)\,\eta(x,-\theta'',\gamma)\\ =-\int_0^\infty d\beta\int_{S^2} d\theta' M(x,\theta',\theta,\alpha-\beta)\,[\eta(x,\theta',\beta)+\mu(x,-\theta',-\beta)\\ +\int_0^\infty d\gamma\int_{S^2} d\theta'' M(x,\theta'',-\theta',\gamma-\beta)\,\eta(x,-\theta'',\gamma)].\end{aligned} \tag{3.75}$$

Define the left-hand side of (3.75) to be $H(x,\theta,\alpha)$, for $\alpha > 0$:

$$\begin{aligned}H(x,\theta,\alpha) := \eta(x,\theta,\alpha)+\mu(x,-\theta,-\alpha)\\ +\int_0^\infty d\gamma\int_{S^2} d\theta'' M(x,\theta'',-\theta,\gamma-\alpha)\eta(x,-\theta'',\gamma), \quad \alpha>0.\end{aligned} \tag{3.76}$$

Then, Equation (3.75) becomes

$$H(x,\theta,\alpha)=-\int_0^\infty d\beta\int_{S^2} d\theta' M(x,\theta',\theta,\alpha-\beta)\,H(x,\theta',\beta),\alpha>0. \tag{3.77}$$

But $H(x,\theta,\alpha)=0$, for $\alpha>0$, is exactly Equation (M')=(3.2), so we can ensure that η will solve (M'), if we require that the only solution of (3.77) be $H=0$. Thus, a sufficient condition that the η obtained under assumptions (M) solves (M') also is that (3.77) has only the trivial solution. We needed the classical conditions on $A(\theta',\theta,k)$ to do the manipulations leading to (3.77), so we include these in our:

ASSUMPTIONS (M').

(a) $A(\theta',\theta,k)$ obeys the classical conditions, (2.31), (2.32) and (2.33);
(b) Equation (3.77) has only the trivial solution $H=0$.

If the conditions (M) and (M') hold, then the unique and miraculous solution to (M) also solves (M'). The function ψ_η, defined by (3.8), is outgoing and has asymptotic behavior determined by the given function $A(\theta',\theta,k)$; but $\psi_\eta=u_q$, the scattering solution to the Schrödinger equation, and so the asymptotic behavior of ψ_η is also determined by $A_q(\theta',\theta,k)$. Thus, $A=A_q$ and the loop is closed. We describe this by the diagram

$$\left.\begin{array}{ll}(M\,and\,M'): & A\to\eta\to\psi_\eta \quad \text{whose asymptotics are A}\\ & \qquad\downarrow \qquad\qquad\qquad\qquad\qquad\qquad ;\psi_\eta=u_q\\ & \qquad q\to u_q \quad \text{whose asymptotics are } A_q\end{array}\right\}\Rightarrow A=A_q$$

or briefly

$$A\to q\to A_q=A.$$

PROOF OF THE IDENTITY (3.72).

$$\begin{aligned}I(x,\theta'',\theta,\alpha,\gamma) := \int_{-\infty}^\infty d\beta\int_{S^2} d\theta' M(x,\theta',\theta,\alpha+\beta)\,M(x,\theta'',-\theta',\beta+\gamma)\\ =\int_{-\infty}^\infty \frac{dk_1}{2\pi}\int_{-\infty}^\infty \frac{dk_2}{2\pi}\int_{S^2} d\theta'\int_{-\infty}^\infty d\beta\,\frac{ik_1}{2\pi}\,A(\theta',\theta,k_1)e^{ik_1(\theta'-\theta)\cdot x-ik_1(\alpha+\beta)}\end{aligned} \tag{3.78}$$

$$\times \frac{ik_2}{2\pi} A(\theta'', -\theta', k_2) e^{ik_2(\theta''+\theta')\cdot x - ik_2(\beta+\gamma)}. \tag{3.79}$$

The integral over β gives $2\pi\ \delta(k_1 + k_2)$. Thus,

$$\begin{aligned} I(x, \theta'', \theta, \alpha, \gamma) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{S^2} d\theta' \frac{ik_1}{2\pi} \left(-\frac{ik_1}{2\pi}\right) A(\theta', \theta, k_1) A(\theta'', -\theta', -k_1) \\ \times e^{ik_1(\theta'-\theta-\theta''-\theta')\cdot x}\, e^{-ik_1(\alpha-\gamma)}. \end{aligned} \tag{3.80}$$

Using reality and reciprocity we get for the right hand side,

$$\int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \left(-\frac{ik_1}{2\pi}\right) e^{ik_1(-\theta-\theta'')\cdot x - ik_1(\alpha-\gamma)} \left[\frac{ik_1}{2\pi} \int_{S^2} d\theta' \overline{A(\theta', -\theta'', k_1)} A(\theta', \theta, k_1)\right],$$

and using unitarity we get

$$I(x, \theta, \theta'', \alpha, \gamma) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \left(-\frac{ik_1}{2\pi}\right) e^{ik_1(-\theta-\theta'')\cdot x - ik_1(\alpha-\gamma)} \left[A(-\theta'', \theta, k_1) - \overline{A(\theta, -\theta'', k_1)}\right] \tag{3.81}$$

$$\begin{aligned} &= -M(x, -\theta'', \theta, \alpha-\gamma) - M(x, \theta, -\theta'', \gamma-\alpha) \\ &= -M(x, -\theta'', -\theta, \gamma-\alpha) - M(x, -\theta'', \theta, \alpha-\gamma) \end{aligned} \tag{3.82}$$

4. MISCELLANEOUS REMARKS

4.1. Comparison with Other Approaches

A complete characterization of the scattering data for the 3D scattering problem has been given in [3] (see also [4,12, p. 241,13,14]. The logical structure of this characterization is the following: given $A(\theta', \theta, k)$, one writes an integral equation whose kernel is defined by $A(\theta', \theta, k)$; if this equation has a solution with certain properties (these properties are algorithmically verifiable given a solution) then: a) $A(\theta', \theta, k) = A_q(\theta', \theta, k)$, where $A_q(\theta', \theta, k)$ is the scattering amplitude corresponding to a local potential from a given class (for instance, $Q(\beta)$); b) the potential $q(x)$ is uniquely defined by the solution; and c) there is, at most, one solution with the above properties. This characterization has one drawback: there is no, in general, algorithm given to check directly from the data if the above integral equation indeed has the solution with the required properties. This drawback is present in the more complicated characterization obtained in [1]. We do not describe their characterization in detail, since it would require much introductory material which is not relevant to our presentation.

Generally speaking, the characterization of the scattering data for potentials in the Schwartz class given in [1] is of the following nature. Given the data $A(\theta', \theta, k)$, one writes an integral equation whose kernel is defined by the data. If this integral equation has a solution with a number of properties (which include the requirement that certain functions related to the solution admit analytic continuation to some noncompact manifolds and satisfy certain differential equations and other requirements), then the given function is a scattering amplitude corresponding to a potential from the Schwartz class. No conditions are given in [1] directly on the data $A(\theta', \theta, k)$ for the above integral equation to have a solution with the needed properties. The properties themselves are more difficult to check than the properties of the solution in [3], where the properties are easily algorithmically verifiable. On the other hand, the basic integral equation in [3] is nonlinear, while the basic equation in [1 (e.g., (1.7) on p. 95)] is linear. The works of many authors (including Faddeev [15], Newton [2], Ablowitz and Nachman [16]) contain necessary conditions but not a characterization (i.e., both necessary and sufficient conditions) on the scattering data to be the scattering amplitude for a local potential from a given class. In [14], an algorithmically verifiable characterization for small potentials is given. In this result, the crucial point that made it possible to give an algorithmically verifiable characterization was the fact that for "small" (in an appropriate sense) data the basic integral equation used in [3] has precisely one solution, so that it only remains to check that this solution has the required properties, which is possible to do algorithmically. (See also Lemma 2.4.3 of [2].)

On page 63 of [2], it is stated that the characterization of the scattering data obtained in [4] is "of limited utility." (Theorem 2.4.5 of [2] misstates this characterization: the last sentence of Theorem 2.4.5 in [2] should contain "... if (B) holds ..." in place of " ... if (A) holds ..."). The reason given in [2] for the conclusion concerning "limited utility" of the characterization obtained in [4] is that, if the conditions (B) of Theorem 2.4.5 in [2] are satisfied, then one can just check if $A_q(\theta', \theta, k)$ (obtained from the constructed q) equals the data $A(\theta', \theta, k)$ or not. This point we have already discussed in Section 3.3. The author of [2, p. 63] writes that "our aim, instead, is to find conditions to ensure the closing of the circle $A \rightarrow q \rightarrow A$ that refer directly to the input function $A(\theta', \theta, k)$." While this aim (which is to find a sufficient condition for closing the above circle) is certainly a worthwhile one (and we pursued this aim in Section 3.3) the stated sufficient condition (Theorem 2.4.7) is not competely proven in [2] because of the errors in the proofs mentioned above. Moreover, the sufficient conditions in Theorem 2.4.7 in [2] require to check whether the solution to Equation (M) is miraculous. This step is not algorithmically verifiable in terms of the data $A(\theta', \theta, k)$. In principle, this step has no advantages over the checking that $A_q(\theta', \theta, k) = A(\theta', \theta, k)$ or over the checking if the solution satisfies the conditions used in [3] or [4]. In the recent paper [5], a number of interesting results are formulated, that may help to fill gaps in the scheme used in [2] for closing the loop $A \rightarrow q \rightarrow A$. Theorem 5.1 in [5] formulates a new characterization of the scattering data for the potentials for which $k = 0$ is not an eigenvalue or resonance (half-bound state). The class of the reconstructed potentials is not specified. In [5], the inverse scattering problem is considered without the assumption that $q(x)$ produces no bound state. We make this assumption for simplicity. As we pointed out in the Introduction, there is an error in the proof of crucial Lemma 4.3, in p. 2423 of [5]. Hopefully, the modified uniqueness result for the problem

$$\Delta u - u_{tt} = q(x)u,\ t > -|x|,\ x \in \mathbf{R}^3,\ u(x,0) = 0,\ u(x, -|x|) = 0,\ u(x,t) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad (4.1)$$

can be proved. For $q(x) = 0$, we have proved that (4.1) has only the trivial solution.

We emphasize the logical structure of our argument: Step 1: given $A(\theta', \theta, k)$, which satisfies conditions M, we derive existence and uniqueness of local real-valued $q(x)$ to which the scattering amplitude $A_q(\theta', \theta, k)$ corresponds; additional assumptions (M') are needed for Step 2, in which we prove that $A_q(\theta', \theta, k) = A(\theta', \theta, k)$. This Step 2 closes the loop $A \rightarrow q \rightarrow A$. Without it, the inverse problem is not solved. An extensive discussion of this point in the one-dimensional case is given in [17].

4.2. Counter Example

The uniqueness statement in [5, p. 2423] is this: if $q(x)$ has no bound states, then $f = 0$ is the only solution of

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) f(x,t) = q(x) f(x,t), \quad x \in \mathbf{R}^3, -|x| \le t, \quad (4.2)$$

$$f(x, t = -|x|) = 0, \quad (4.3)$$

$$\lim_{|x| \rightarrow \infty} f(x,t) = 0, \quad (4.4)$$

where t in (4.4) is arbitrary but fixed, $t > -|x|$. Our counter example is this. Take $q(x) = 0$, which clearly has no bound states, then let $f(x,t)$ be

$$f(x,t) = \begin{cases} f_1(x,t) = 0 \text{ for } x,t \in \{-t > |x|\}, \text{ the backward light cone,} \\ f_2(x,t) = \frac{\phi(|x|+t)}{|x|} \text{ for } x,t \in \{|x| > |t|\}, \text{ outside both light cones,} \\ f_3(x,t) = \frac{\phi(|x|+t)+\phi(|x|-t)}{|x|} \text{ for } x,t \in \{t > |x|\}, \text{ the forward light cone,} \end{cases}$$

where $\phi(y)$ is an odd function on $\mathbf{R}$, infinitely differentiable with compact support and vanishing identically in a neighborhood of $y = 0$. It is clear that $f(x,t)$ satisfies the wave equation, $f(x,t = -|x|) = 0$ and (4.4) holds. Its smoothness across the light cones and on $|x| = 0$ is also clear because $\phi(y)$ vanishes in a neighborhood of zero, so near the backward light cone, $t = -|x|$, $f_2(x,t) = 0$ as does f_1, and near the forward light cone, $t = |x|$, $\phi(|x| - t) = 0$, so $f_3(x,t) = (\phi(|x| + t)/(|x|) = f_2(x,t)$. At $|x| = 0$ in the forward light cone, $f_3(x,t)|_{x=0} = \phi'(t) + \phi'(-t)$, (because $\phi(|x| + t) + \phi(|x| - t) = 0$ at $|x| = 0$, since $\phi(y)$ is odd). Thus, $f(x,t)$ is a smooth function everywhere.

4.3 Open Problems

An important open problem is to give conditions directly on the data $A_q(\theta',\theta,k)$, which are sufficient for $q(x)$ to have some properties. There are very few results of this nature: in [18–21], it is proved that $q(x) = q(|x|)$, if $A(\theta',\theta,k) = F(\theta'.\theta,k)$, for all $\theta',\theta \in S^2$, and one fixed k, if $q \in Q_a$, or at all $k > 0$, if $q \in Q(\beta)$. It is noted in [20] that for real c, $cA_q(\theta',\theta,k)$ is not a scattering amplitude unless $c = 0$ or 1 for $q \in Q_a$. For example, $-A_q(\theta',\theta,k)$ is not a scattering amplitude. In [2], it is noted that $\overline{A_q(\theta',\theta,k)}$ is not a scattering amplitude (as is obvious from unitarity). It is conjectured in [20] that $A_q(\theta',\theta,k)$ cannot be a finite rank operator in $L^2(S^2)$ for $q \in Q_a$. It is proved in [8] that $A_q(\theta',\theta,k) = c \neq 0$, for one fixed $k > 0$, cannot be the scattering amplitude for $q \in Q_a$. It is conjectured in [22] that, if $q \in Q_a$, then the smallest possible value for a is

$$a = \limsup_{\ell\to\infty} \left\{ [\sup_{\substack{\alpha\in S^2 \\ -\ell\leq m\leq \ell}} |A_{\ell m}(\alpha)|]^{2/(2\ell+1)} \frac{2\ell+1}{e} \right\}.$$

If $q(x)$ is small, in the sense that the operator in $L^2(S^2)$ with the kernel $(ik)/(2\pi)\ A(\theta',\theta,k)$, for all $k \in (0,\infty)$, has norm less than 1, then the S_x operator in $L^2(S^2)$ has a canonical factorization. This allows one to give an algorithmically verifiable characterization of the scattering data in the 3D inverse scattering problem which we study here [14].

Much is not known. For example, no directly verifiable conditions on $A_q(\theta',\theta,k)$ are known for $q(x)$ to be non-negative; or for $q(x)$ to belong to some locally smooth class of functions, say $C^m_{\text{loc}}(\mathbf{R}^3)$, the space of m times continuously differentiable functions, Q or for $q(x)$ to have a prescribed rate of decay, e.g., $|q(x)| \leq C(1 + |x|)^{-\beta}$ for $x \to \infty$.

5. STATEMENT OF THE RESULTS

We can now formulate the basic results obtained in Sections 2 and 3. Recall that Assumptions (M) are formulated below in formula (3.16), and (M') below in (3.77).

THEOREM 1. *Assume that the function $A(\theta',\theta,k)$ is such that Assumptions (M) hold. Then there is a unique real-valued local potential $q(x)$ constructed from the function $A(\theta',\theta,k)$ by formula (3.4). The function ψ_η, defined in equation (3.6), solves the Schrödinger equation*

$$[\nabla^2 + k^2 - q(x)]\psi_\eta = 0 \quad \textit{in } \mathbf{R}^3$$

with this $q(x)$ and has the asymptotics

$$\psi_\eta = \exp(ik\theta\cdot x) + A_q(\theta',\theta,k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right),\ r = |x| \to \infty,\ \frac{x}{r} = \theta'.$$

THEOREM 2. *Assume that the function $A(\theta',\theta,k)$ is such that Assumptions (M) and (M') hold, Then the potential $q(x)$, constructed in Theorem 1, produces the scattering data $A_q(\theta',\theta,k)$ identical to the data $A(\theta',\theta,k)$, with which we started.*

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